11. Cont:
In collection: Numerical Methods for Solving Problems of Mathematical Physics, Vol.6, №4, "Nauka", Moscow, 1966.

Translated by J. J. D.

UDC 532.51

NOTES ON THE THREE-DIMENSIONAL FLOW PATTERN OF A PERFECT FLUID IN THE PRESENCE OF A SMALL PERTURBATION OF THE INITIAL VELOCITY FIELD

PMM Vol.36, №2, 1972, pp.255-262
V. I. ARNOL'D
(Moscow)
(Received March 31, 1971)

Euler equations of the three-dimensional motion of a perfect incompressible fluid, linearized for a nearly stationary flow are considered and the class of stationary flows for which these linearized equations admit exact explicit solutions is indicated. The analysis of derived equations shows that in some stationary flows the perturbation buildup considerably differs from that obtaining in cases generally considered in the theory of hydrodynamic stability: there appears an infinitely great number of unstable configurations, the flow pattern is difficult to predict (since an approximate determination of perturbation development with time necessitates a rapidly increasing amount of information about initial conditions, etc). These differences are due to the different geometry of stationary flows. In the recently constructed models of stationary flows the assumption is made that a fluid particle in motion stretches into a filament or ribbon whose length exponentially increases with time, while in the usually considered flows the length is assumed to be a linear function of time. In two-dimensional flows the phenomenon of exponential stretching of particles is impossible. It is shown that this is, also, impossible in three-dimensional flows in which the vectors of velocity and viscosity are not collinear.

1. The linearized Euler equation. The shortened equation.

Let us write Euler's equation in the form of a vortex equation

$$\partial \mathbf{r} / \partial t = \{\mathbf{v}, \mathbf{r}\} \qquad (\mathbf{r} = \operatorname{rot} \mathbf{v}) \tag{1.1}$$

where the Poisson's bracket of the two vector fields is defined by the condition

$$D_{\{\mathbf{a},\mathbf{b}\}} = D_\mathbf{b} D_\mathbf{a} - D_\mathbf{a} D_\mathbf{b}$$

in which $D_\mathbf{q}$ denotes integration in the direction of field $\mathbf{q}$. Let us consider a small perturbation $\mathbf{u}$ of the stationary flow $\mathbf{v}$. Let $\mathbf{s}$ be the vortex perturbation field: $\operatorname{rot}(\mathbf{v} + \mathbf{u}) = \mathbf{r} + \mathbf{s}$. Equation (1.1) linearized in the neighborhood of flow $\mathbf{v}$ is of the form

$$\partial \mathbf{s} / \partial t = \{\mathbf{v}, \mathbf{s}\} + \{\operatorname{rot}^{-1} \mathbf{s}, \mathbf{r}\} \tag{1.2}$$

The operation rot^{-1} is understood as the restitution of a nondivergent vector field over its vortex field. In the multiply-connected case it is necessary to consider instead of the vortex field the totality of velocity field circulations over all possible closed contours (not necessarily homologous to zero), i.e., the vortex field together with velocity field circulations over basic univariate cycles. If the flow region has a rim, the velocity field is to be assumed tangent to it.

Let us examine the bahavior of solutions of this equation which is linear with respect to s. Note that the first term in the right-hand side of (1.2) is a more powerful linear operator over s than the second. Hence the second term may be considered as a perturbation of the first. In this way we obtain the shortened equation

$$\partial \mathrm{s} / \partial t = \{\mathrm{v}, \mathrm{s}\} \tag{1.3}$$

If the stationary flow is potential ($\mathbf{r} = 0$), the second term in Eq. (1.2) vanishes, and in that case the shortened equation (1.3) is the same as the linearized Euler equation (1.2). In accordance with the theory of perturbations [1] it is reasonable to assume that the shortened equation defines the continuous part of the spectrum of Eq. (1.2).

The shortened equation (1.3) implies that vector s is carried by the stationary flow. If the geometry of the stationary flow v is known, this equation can be solved explicitly. Let $\{g^t\}$ be a one-parameter group of diffeomorphisms induced by the stationary flow, hence $g^t\,(x)$ is the solution of the system of ordinary differential equations

$$\frac{d}{dt}\, g^t(x) = \mathrm{v}\,(g^t(x)), \qquad g^0(x) = x \tag{1.4}$$

Solution for s of the shortened equation (1.3) can now be expressed in terms of its initial conditions by formula

$$\mathrm{s}\,(t, x) = g_*{}^t \mathrm{s}\,(0, g^{-t}(x)) \tag{1.5}$$

where $g_*{}^t$ is the derivative of the image of g^t.

2. The action - angle variables. The geometry of stationary flows of perfect fluid was examined in [2]. It is shown there that, when the fields v and r are not identically collinear in any region, the space filled with fluid becomes divided into cells in each of which the stream- and vorticity-lines lie on torus surfaces (*). Curvilinear coordinates, similar to the action – angle variables in conventional mechanics. We denote these coordinates by φ and z. The coordinates $\varphi = (\varphi_1, \varphi_2) \bmod 2\pi$ are angular coordinates along the tori and z is the "action variable" numbering the latter. The coordinates φ_1, φ_2 and z can be chosen so that a volume element is defined by $d\varphi_1 d\varphi_2 dz$ and the fields v and r by

$$\mathrm{v} = v_1(z)\frac{\partial}{\partial\varphi_1} + v_2(z)\frac{\partial}{\partial\varphi_2}, \qquad \mathbf{r} = r_1(z)\frac{\partial}{\partial\varphi_1} + r_2(z)\frac{\partial}{\partial\varphi_2}$$

Equations (1.4) are integrable in the system of coordinates φ, z. For the components of field s in coordinates φ, z

$$\mathrm{s}\,(t; \varphi, z) = s_1\frac{\partial}{\partial\varphi_1} + s_2\frac{\partial}{\partial\varphi_2} + s_3\frac{\partial}{\partial z}$$

from (1.5) we obtain the expressions

---

*) Cells of a different kind in which all streamlines are closed are also possible in the case of flow in a manifold with a rim; this case is not considered here.

$$s_{1,2}(t;\varphi,z)=s_{1,2}(0;\varphi_0,z)+tv_{1,2}s_3(0;\varphi_0,z),\ s_3(t;\varphi,z)=s_3(0;\varphi_0,z)$$
$$(\varphi_0=\varphi-\nu t) \tag{2.1}$$

where the prime denotes a derivative with respect to z.

Formulas (2.1) imply that solutions of the shortened equation (1.3) (for $\mathrm{v}'\neq 0$) usually increase linearly with time. Hence the conventional (exponential) instability of linearized Euler equation can only be due to the second term in formula (1.2). In accordance with the theory of perturbations it is reasonable to expect the appearance of a finite number of unstable discrete eigenvalues (there is no rigorous proof of this).

An interesting exception is the instability of the Couette flow between two cylinders (this was brought to the author's attention by V. I. Iudovich).

In a Couette flow the velocity component of the basic flow along the cylinder axis is zero, hence invariant with respect to Bernoulli's constant. This results in the degeneration of a whole segment of the continuous spectrum into a single point. The longitudinal velocity component $v_{\mathbf{m}}$ in formula (3.1) is for certain values of the wave vector $\mathbf{m}$ independent of z.

The assumption of finiteness of the number of unstable configurations relates to a nondegenerate continuous spectrum, when the longitudinal velocity component varies with Bernoulli's constant, i.e., $v_{\mathbf{m}}'\neq 0$ in (3.1). For such nondegeneracy to exist it is, for example, sufficient for the curvature of the plane curve $v_1=v_1(z)$, $v_2=v_2(z)$ to be nonzero and for the curve to be regular.

The question of retention of the detected above slow instability when passing from the shortened equation (1.3) to the complete equation (1.2) is discussed in Sect. 4 below. The other possibility of exponential instability is related to the collinearity of v and $\mathbf{r}$, when the action – angle variables cannot be introduced and the stationary flow geometry differs from the described above (cf. [3]). This form of instability is examined in Sect. 5.

3. Spectrum of the shortened equation. For a more detailed analysis of solutions of Eq. (1.3) we expand s into a Fourier series in terms of φ, using the following notation. Let $\mathbf{m}$, which we shall call the wave vector, be a pair of integers m_1 and m_2. We denote $m_1\varphi_1+m_2\varphi_2$ by $(\mathbf{m},\ \varphi)$ the number $\sqrt{m_1^2+m_2^2}$ by m and the pair $n_1=-m_2$ and $n_2=m_1$ by $\mathbf{n}$.

For each wave vector we determine the "longitudinal", "transverse", and "normal" vector fields

$$\mathbf{e}_{\mathbf{m}}=\frac{m_1}{m}\frac{\partial}{\partial\varphi_1}+\frac{m_2}{m}\frac{\partial}{\partial\varphi_2},\qquad \mathbf{e}_{\mathbf{n}}=-\frac{m_2}{m}\frac{\partial}{\partial\varphi_1}+\frac{m_1}{m}\frac{\partial}{\partial\varphi_2},\qquad \mathbf{e}_z=\frac{\partial}{\partial z}$$

(For $m=0$ we assume, e.g., $\mathbf{e}_{\mathbf{m}}=\partial/\partial\varphi_1$ and $\mathbf{e}_{\mathbf{n}}=\partial/\partial\varphi_2$).

The Fourier expansion of field s can now be written as

$$\mathrm{s}=\sum_{\mathbf{m}}(A_{\mathbf{m}}\mathbf{e}_{\mathbf{m}}+B_{\mathbf{m}}\mathbf{e}_{\mathbf{n}}+C_{\mathbf{m}}\mathbf{e}_z)\,e^{i(\mathbf{m},\varphi)}$$

where $A_{\mathbf{m}}$, $B_{\mathbf{m}}$ and $C_{\mathbf{m}}$ are functions of z.

It can be readily verified that the divergence of fields $\mathbf{e}_{\mathbf{m}}$, $\mathbf{e}_{\mathbf{n}}$ and $\mathbf{e}_z$ is zero (this is the result of the form $d\varphi_1\,d\varphi_2 dz$) of the volume element). Hence

$$\operatorname{div}\mathrm{s}=\sum_{\mathbf{m}}(imA_{\mathbf{m}}+DC_{\mathbf{m}})\,e^{i(\mathbf{m},\varphi)}\qquad\left(D=\frac{d}{dz}\right)$$

Consequently the nondivergent fields are determined by the condition " $imA_{\mathbf{m}}+$

$+ DC_{\mathbf{m}} = 0$ for all $\mathbf{m}$".

In accordance with this condition the set of functions $B_{\mathbf{m}}$ and $C_{\mathbf{m}}$ (for $\mathbf{m} = 0$ we have $C_0 = \text{const}$ but A_0 is to be added) can be taken as the "coordinates" in the space of convergent fields. In such system of coordinates Eq. (1.3) is decomposed into a series of triangular systems

$$\begin{cases} \dot{B}_{\mathbf{m}} = -imv_{\mathbf{m}}B_{\mathbf{m}} + v_{\mathbf{n}}'C_{\mathbf{m}} \\ \dot{C}_{\mathbf{m}} = -imv_{\mathbf{m}}C_{\mathbf{m}} \end{cases} \tag{3.1}$$

where $\mathbf{v} = v_{\mathbf{m}}\mathbf{e}_{\mathbf{m}} + v_{\mathbf{n}}\mathbf{e}_{\mathbf{n}}$ is the velocity field of the stationary flow (for $\mathbf{m} = 0$ we add the equation $\dot{A}_0 = v_0' C_0$); the prime and the dot denote differentiation with respect to z and t respectively.

Formula (3.1) again implies the nonexponential instability of Eqs. (1.3). Furthermore, it contains the definition of the spectrum of Eq. (1.3): to each wave vector $\mathbf{m}$ corresponds a segment of the continuous spectrum along the imaginary axis. The related "frequencies" $mv_{\mathbf{m}}$ are equal to all kinds of frequencies $(\mathbf{m}, \mathbf{v})$ of the stationary flow at various tori corresponding to various values of the z-coordinate. The multiplicity of each segment is not less than two (the B- and C-components have the same frequencies).

4. The theorem of Squire for shear flows. The coordinates introduced above are suitable for analyzing the shortened equation (1.3), however, since in curvilinear coordinates the operator rot^{-1} is of a complicated form, analysis of the complete equation (1.2) is generally difficult. A particular case in which the analysis can be reduced to a one-dimensional problem is that of flow with straight streamlines. All plane rectilinear flows, as well as the more general ones in which the fluid flows in parallel planes at constant velocity which varies in magnitude and direction when passing from one plane to another, belong to this class. Study of the latter may be considered as an approximate analysis of general flow in the torus geometry, in which the torus curvature is neglected, while shear (variation of the direction of streamlines from one torus to another) is taken into consideration.

Let φ_1, φ_2 and z be Cartesian coordinates and $dl^2 = d\varphi_1^2 + d\varphi_2^2 + dz^2$. In this case it is expedient to consider periodic flows of not necessarily 2π periodicity (e.g. we can assume the periods of φ_1 and φ_2 to be $2\pi X_1$ and $2\pi X_2$, respectively). The only alteration to be introduced in formulas in Sect. 3 is that now the wave vector $\mathbf{m}$ does not run through a grid of whole points but through grid $\{(m_1 / X_1, m_2 / X_2)\}$.

On these assumptions the expansion of the vortex field $\mathbf{r}$ in terms of unit vectors $\mathbf{e}_{\mathbf{m}}$, $\mathbf{e}_{\mathbf{n}}$ and $\mathbf{e}_z$ is of the form $\mathbf{r} = -v_{\mathbf{n}}'\mathbf{e}_{\mathbf{m}} + v_{\mathbf{m}}\mathbf{e}_{\mathbf{n}}$. The matrices of operator rot in coordinates $B_{\mathbf{m}}$, $C_{\mathbf{m}}$ and the operator of Poisson's bracket containing $\mathbf{r}$ are, respectively,

$$im\begin{pmatrix} 0 & -E + m^{-2}D^2 \\ E & 0 \end{pmatrix}, \quad -\begin{pmatrix} imv_{\mathbf{n}}' & v_{\mathbf{m}}'' \\ 0 & imv_{\mathbf{n}}' \end{pmatrix}$$

where E is an identical transformation. Hence in our coordinates the linearized Euler's equation (1.2) is decomposed into a set of systems of equations corresponding to various $\mathbf{m}$. After calculation we obtain for $\mathbf{m} \neq 0$ the triangular system

$$\begin{aligned} \dot{B}_{\mathbf{m}} &= \left[imv_{\mathbf{m}} + \frac{v_{\mathbf{m}}''}{im}(E - m^{-2}D^2)^{-1} \right] B_{\mathbf{m}} \\ \dot{C}_{\mathbf{m}} &= imv_{\mathbf{m}}C_{\mathbf{m}} + v_{\mathbf{n}}'(E - m^{-2}D^2)^{-1}B_{\mathbf{m}} \end{aligned} \tag{4.1}$$

and for $\mathbf{m} = 0$ we have the system $\dot{A}_0 = \dot{B}_0 = \dot{C}_0 = 0$. The first equation becomes separated and, if the B-component does not have exponential instability, the latter is also absent in the C-component (this is implied by the nonhomogeneous linear equation obtaining for C_m).

Note that the equation for B_m contains only the longitudinal velocity component v_m. Hence this equation is the same as that derived in the analysis of two-dimensional flow of a perfect fluid, whose velocity profile is the component $v_m(z)$ of the velocity vector of a three-dimensional flow in the direction of the wave vector $\mathbf{m}$.

Thus a rectilinear three-dimensional flow is exponentially unstable when, and only when, at least one of the two-dimensional flows of a perfect fluid, obtained by the substitution for the velocity vector $\mathbf{v}$ of its longitudinal component v_m, is exponentially unstable. The problem of exponential instability of the considered class of three-dimensional flows of perfect fluid is thus reduced to a similar problem for a series of two-dimensional flows corresponding to various values of the wave vector.

In the particular case of flow free from shear (constant direction of $\mathbf{v}$) all velocity profiles are proportional to each other and the obtained result conforms to the theorem of Squire for a perfect fluid [4].

The Jordan form of system (4.1) tends to indicate that in three-dimensional flows, unlike in two-dimensional ones, the linear increase of vortex perturbations with time is the rule, even in the absence of exponential instability.

5. Stationary flows with exponential stretching of particles. The flow region is a three-dimensional compact manifold M constructed in the following manner. (*)

Let us, first, consider a conventional three-dimensional space in coordinates x, y, z and determine the following three diffeomorphisms of that space:

$$T_1(x, y, z) = (x+1, y, z), \quad T_2(x, y, z) = (x, y+1, z)$$

$$T_3(x, y, z) = (2x+y, x+y, z+1)$$

Each of these transforms into itself the lattice of points with complete coordinates x, y, z. Let us identify all points of the $x\,y\,z$-space which can be obtained from each other by successive application of T_i and T_i^{-1} (in any order). As the result a compact analytic manifold M is created which may be visualized as the product of multiplication of a two-dimensional torus $\{(x, y) \bmod 1\}$ by segment $0 \leqslant z \leqslant 1$ whose end tori are identified by formula $(x, y, 0) \equiv (2x+y, x+y, 1)$. We introduce on the drived manifold a Riemann metric. For this we construct in the $x\,y\,z$-space a Riemann metric invariant with respect to all T_i.

Let us examine the linear transformation of the $x\,y$-plane

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ x+y \end{pmatrix}, \qquad (A) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Transformation A has eigenvalues $\lambda_{1,2} = (3 \pm \sqrt{5})/2$. Note that $\lambda_1 > 1 > \lambda_2 > 0$, $\lambda_1\lambda_2 = 1$, and the eigendirections are orthogonal to each other. Let (p, q)

*) This kind of manifold became important in the contemporary qualitative tneory of ordinary differential equations after the work of Smile, whose attention was drawn to this example by Thom.

be a Cartesian system of coordinates in the $x\ y$ -plane with the axes p and q directed along eigenvectors with eigenvalues $\lambda_1 > 1$ and $\lambda_2 < 1$, respectively.

Let us set
$$ds^2 = e^{-2\mu z}dp^2 + e^{2\mu z}dq^2 + dz^2 \quad (\mu = \ln \lambda_1) \tag{5.1}$$

Metric ds^2 is invariant with respect to transformations T_i, hence it defines on the three-dimensional compact manifold M an analytic Riemann structure.

Let us now consider the vector field $\partial / \partial z$ in the xyz-space. Since it is invariant with respect to transformations T_i, it defines the vector field v on manifold M. Field v on the Riemann manifold M is harmonic: div v $= 0$, rot v $= 0$. Hence **v** may be taken as the velocity field of a stationary potential flow of a perfect fluid. Every particle of fluid moving in that field exponentially stretches in the q-direction and contracts in the p direction, as implied by formula (5.1).

6. Analysis of the linearized Euler equation. Since the considered flow is potential, the linearized Euler's equation (1.2) is equivalent to the shortened equation (1.3). Owing to the simple geometry of flow the latter equation is solved by formula (1.5). The solution is conveniently expressed with the use of the following notation. Let us consider in the pqz-space the vector fields

$$\mathbf{e}_p = e^{\mu z}\frac{\partial}{\partial p}, \qquad \mathbf{e}_q = e^{-\mu z}\frac{\partial}{\partial q}, \qquad \mathbf{e}_z = \frac{\partial}{\partial z}$$

These fields are invariant with respect to all transformations T_i, consequently they can be considered as vector fields on the manifold M. The directions of fields $\mathbf{e}_p$, $\mathbf{e}_q$ and $\mathbf{e}_z$ are invariant with respect to the phase stream g^t of field $\mathbf{e}_z$ (in the coordinate form $g^t\ (p,\ q,\ z) = (p,\ q,\ z + t)$). Under the action of the stream the fields themselves are transformed by formulas

$$g_*{}^t\mathbf{e}_p = e^{-\mu t}\mathbf{e}_p, \qquad g_*{}^t\mathbf{e}_q = e^{\mu t}\mathbf{e}_q, \qquad g_*{}^t\mathbf{e}_z = \mathbf{e}_z$$

In accordance with this, the direction of field $\mathbf{e}_q$ is called the stretching direction, that of $\mathbf{e}_p$ the compression direction, and that of $\mathbf{e}_z$ the neutral direction. Any vector field w on M can be decomposed in these directions

$$\mathrm{w} = w_p\mathbf{e}_p + w_q\mathbf{e}_q + w_z\mathbf{e}_z$$

where w_p, w_q and w_z are functions on the manifold M.

Formula (1.5) applied to the stationary flow v $= \mathbf{e}_z$ has with the introduced notation the form

$$s_p\,(t) = e^{-\mu t}U^t s_p\,(0), \qquad s_q\,(t) = e^{\mu t}U^t s_q\,(0), \qquad s_z\,(t) = U^t s_z\,(0) \tag{6.1}$$

where U^t is a linear operator acting on functions on the manifold M by formula "$(U^t\ f)$ $(\xi) = f\ (g^{-t}\xi)$ for any point ξ from M". Note that the stream g^t maintains its volume, hence operator U^t is unitary.

Formula (6.1) provides fairly complete answers to all kinds of questions on the growth of perturbations of a stationary flow v. First, it shows that the q-component of vortex perturbation exponentially increases with time, while the p component is exponentially attenuated.

Next, the spectrum of operator U^t can be easily analyzed by a Fourier series expansion in terms of $(x,\ y)$ with fixed z, and for functions independent of x and y by such expansion in terms of z. This spectrum has a countably-multiple continuous (Lebesgue) component along a unit circle and, also, a discrete set of eigenvalues corresponding to eigenfunctions $\varphi_m = e^{2\pi imz}$ (m are integers). This implies that Euler's equation (1.2) linearized for a close to stationary flow $\mathbf{v} = \mathbf{e}_z$ has a countable set of unstable eigenvalues $\mu - 2\pi im$ related to the countable set of increasing perturbations of vortex $\mathbf{s} = \varphi_m(z)\,\mathbf{e}_q$ $(m = \pm 1, \pm 2, \ldots)$.

The difficulty of predicting solutions of the linearized Euler equation (1.2) for flows with exponential stretching of particles is also indicated by formulas (6.1): to find an approximate solution in terms of t it is necessary to know with considerable exactitude a number of high order harmonics in the initial perturbation $s(0)$ which rapidly increase with t. Comparison of formulas (6.1) and (2.1) shows that the exponential increase of particle stretching considerably increases the difficulty of predicting the growth of perturbations, as compared with conventional flows with linear stretching of particles considered in Sect. 2 – 4.

Phenomena similar to those brought to light in this example are to be expected also in other flows with exponentially stretched particles, and such flows are possible in regions of conventional three-dimensional space. Experimental confirmation of this can be found in [5, 6]. Computer calculations cited in [5] tend to show that the stationary flow of a perfect fluid specified by formulas [2]

$$v_x = A\sin z + C\cos y, \qquad v_y = B\sin x + A\cos z, \qquad v_z = C\sin y + B\cos z$$

has the property of exponentially stretched particles.

The author thanks L. D. Faddeev and V. I. Iudovich for valuable discussions.

BIBLIOGRAPHY

1. Faddeev L. D., On the theory of stability of plane-parallel stationary flows of perfect fluid. Boundary Value Problems of Mathematical Physics, Vol. 5, Zap. Nauchn. Seminarov LOMI, Vol. 21, pp. 164-172, "Nauka", Moscow, 1971.

2. Arnold V., Sur la géométrie differentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. Annales de l'Institut Fourier, Vol. 16, №1, p. 347, 1966.

3. Grad H., Mathematical problems arising in plasma physics. Actes du congrès international des mathématiciens. Vol. 3, pp. 105-113, Gautier-Villars, Paris, 1971.

4. Squire H. B., On the stability of the three-dimensional disturbances of viscous flow between parallel walls. Proc. Roy. Soc., A, Vol. 142, 1933.

5. Henon M., Sur la topologie des lignes de courant dans un cas particulier. Comptes Rendues de l'Acad. des Sc., Vol. 262, Paris, 1966.

6. Froeschle C., A numerical study of the stochasticity of dynamical systems with two degrees of freedom. Astronom. and Astrophys., Vol. 9, 1970.

Translated by J. J. D.